OMO IJS

OMO International Journal of Sciences

Available online at: https://survey.amu.edu.et/ojs/index.php/OMOIJS/issue/archive

Vol. 4 Issue. 1, June, 2021, pages: 76~90

ISSN(Print): 2520-4882 :ISSN(Online):2709-4596

Full-Length Research Article

Fourth-Order Compact Finite Difference Method Combined with Richardson Extrapolation for One-Dimensional Heat Equation

Hailu Muleta Chemeda *   Feyisa Edosa Merga

[a] *Department of Mathematics, Jimma University, Jimma, P.O.Box 378, Ethiopia*
*Contact address: muletah@gmail.com, feyisae.2014@gmail.com,

ABSTRACT

In this paper, Fourth-Order Compact Finite Difference method combined with Richardson extrapolation to solve the one-dimensional heat equation has been presented. The method is found to be fourth-order convergent in space and second-order in time variable. The method is also unconditionally stable and consistent for solving a one-dimensional heat equation. When combined with the Richardson extrapolation, the desired level of numerical solutions can be obtained without the requirement of further mesh refinement which claims extra computational time and memory. To validate the applicability of the proposed method, two model examples are considered and solved for different values of spatial and temporal step lengths. The numerical solutions presented in tables and the simulations of the model problems show that the proposed method approximates the exact solution very well. In a nutshell, the proposed method is efficient and capable of solving the one-dimensional heat equation.

Keywords: fourth-order compact scheme, heat equation, order of convergence, Richardson extrapolation

Received: February 05,2021, Accepted: April 26, 2021, Published: June 25,2021

1. INTRODUCTION

Partial differential equations (PDEs) are mathematical equations that play a significant role in modelling physical phenomena that occur in nature (sun et al., 2010). Because of the complex nature of many practical problems, the majority of the solutions of the PDEs are numerical (Chen et al., 2020). The parabolic equation is used to study diffusion and heat conduction problems. The main characteristic of the parabolic equation is that it models a transient state or evolution diffusion process and requires the specification of appropriate boundary conditions at all boundaries and an initial condition at the starting point of the evolution process (Necati et al., 2017). Because of its wider application in many natural sciences and engineering, the numerical scheme that solves parabolic equations often attracts the attention of many researchers (Samarskii, 2001, Dimove et al., 2016). A numerical technique for parabolic problems may fall under the following categories: the finite difference method (FDM), the finite element method (FEM) and the finite volume method (FVM) (Andrea and Vautard, 2001). Due to their easy implementations and computational efficiency, the finite difference methods are popular in solving parabolic equations (Zhang et al., 2012).

Olusegun et al. (2017) used to solve One-Dimensional heat equation using the explicit finite difference method and concluded that the numerical solution is closer to the exact solution if the mesh size is further refined. However, when the mesh size is further refined the problem results in large system of equations which requires a large memory and long computational time. Benyam (2015) applied the explicit finite difference method and finite element method coupled with the Crank-Nicolson scheme for solving the one-dimensional heat equation. Dabral et al. (2011) also used B-spline finite element method to approximate the one-dimensional heat equation and were able to produce an approximate solution that is in good agreement with the exact solution. Hooshmandsl et al. (2012) applied the Chebyshev Wavelets method to solve the one-dimensional heat equation. And the authors claim that the method is found to be accurate (in the sense of L_2 and L_∞ norms), simple and fast. However, the theoretical order of convergence and stability conditions of the method are lacking. Unlike the traditional methods such as FDM, FEM and FVM used to solve the one-dimensional heat equation, Tatari and Dehghan (2010) applied radial basis functions (particularly, Gaussian radial basis functions) to solve one-, two- and three-dimensional heat equations.

The usual FDM shows shortcomings in computational accuracy. One technique to overcome such shortcomings is to refine the mesh which in turn leads to a large system of equations and hence will increase the amount of storage and CPU time (Chen et al., 2020). Another natural way to increase the accuracy and improve the order of convergence of the numerical solution is through the application of the Richardson extrapolation technique (Wang and Zhang, 2009, Gonzalez et al., 2010 and Liao and Sun, 2011). The technique is to approximate the difference method on two or more consecutive grids and then combine the resulting numerical approximations using appropriate weights that depend on the order of the method in consideration (Marchuk and Shaidurov, 1983, Carlos et al., 2013). The third option is to use the compact finite-difference scheme for discretization of the PDEs which result in the ordinary differential equations (ODEs). The time discretization methods such as the Runge-Kutta, Predictor-Corrector, Cubic splines, etc. may be used to fully discretize the resulting ODEs (Chen et al., 2020).

This paper aims to apply the fourth-order compact finite difference method combined with the Richardson extrapolation technique to the one-dimensional heat equations to generate a more accurate numerical solution. The order of convergence and stability conditions of the proposed method are also established.

The rest of the paper is organized as follows: In section 2, mathematical formulation, order of the method, and stability analysis of the proposed method are carried out. The strategy how to combine the Richardson extrapolation with the proposed method is discussed in section 3. Validating the applicability and efficiency of the proposed method via numerical model examples is presented in section 4 and the paper ends with concluding remarks in section 5.

2. MATHEMATICAL FORMULATION, ORDER OF CONVERGENCE AND STABILITY

2.1. Mathematical Formulation

The 1D heat equation with initial and Dirichlet boundary conditions is given by

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2} \qquad (0,1)\times(0,T] \tag{2.1}$$

subject to the initial condition

$$u(x,0) = f(x), \qquad 0 \le x \le 1 \tag{2.2}$$

and the boundary conditions

$$\begin{cases} u(0,t) = g_1(t) \\ u(1,t) = g_2(t) \end{cases}, \ 0 \le t \le T, \tag{2.3}$$

where $\alpha > 0$ is the diffusive (viscous) coefficient, $f(x)$, $g_1(t)$ and $g_2(t)$ are smooth functions on the given domain. Let the solution domain $[0,1]\times[0,T]$ be divided into N and M equal subintervals with uniform mesh sizes h and k respectively such that $x_i = x_0 + ih,\ i = 0(1)N$ and $t_j = t_0 + jk,\ j = 0(1)M$.

From the Taylors series expansion, we have

$$u_{i+1}=u_i+hu_i^{'}+\frac{h^2}{2}u_i^{''}+\frac{h^3}{6}u_i^{'''}+\frac{h^4}{24}u_i^{(4)}+\frac{h^5}{120}u_i^{(5)}+O\left(h^6\right)$$
$$u_{i-1}=u_i-hu_i^{'}+\frac{h^2}{2}u_i^{''}-\frac{h^3}{6}u_i^{'''}+\frac{h^4}{24}u_i^{(4)}-\frac{h^5}{120}u_i^{(5)}+O\left(h^6\right) \quad (2.4)$$

From Eq. (2.4), we obtain the following expressions

$$u_i^{'}=\delta_x u_i-\frac{h^2}{6}u_i^{(3)}-\frac{h^2}{12}u_i^{(5)}+O\left(h^6\right) \quad (2.5)$$

$$u_i^{''}=\delta_x^2 u_i-\frac{h^2}{12}u_i^{(4)}-\frac{h^4}{360}u_i^{(6)}+O\left(h^6\right) \quad (2.6)$$

Using Eq. (2.6) into Eq. (2.1), we get

$$\frac{\partial u_i}{\partial t}=\alpha\left(\delta_x^2 u_i-\frac{h^2}{12}u_i^{(4)}\right)+O\left(h^6\right) \quad (2.7)$$

Differentiating Eq. (2.1) twice with respect to the spatial variable, we have

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial u_i}{\partial t}\right)=\alpha\frac{\partial^4 u_i}{\partial x^4} \quad (2.8)$$

Substituting Eq. (2.8) into Eq. (2.7), we obtain

$$\frac{\partial u_i}{\partial t}=\alpha\left(\delta_x^2 u_i-\frac{h^2}{12}\frac{\partial}{\partial t}\left(\frac{\partial^2 u_i}{\partial x^2}\right)\right)+O\left(h^4\right)=\frac{\partial u_i}{\partial t}=\alpha\left(\delta_x^2 u_i-\frac{h^2}{12\alpha}u_i^{''}\right) \quad (2.9)$$

Using Eq. (2.6) into Eq. (2.9), we obtain

$$\left(1+\frac{h^2}{12}\delta_x^2\right)\frac{\partial u_i}{\partial t}=\alpha\delta_x^2 u_i+O(h^4) \quad (2.10)$$

The Crank-Nicolson discretization of the system of ode in Eq. (2.10) at a time level j can be expressed as:

$$\frac{u_i^{j+1}-u_i^j}{k}\left(1+\frac{h^2}{12}\delta_x^2\right)=\frac{\alpha\delta_x^2}{2}\left(u_i^{j+1}+u_i^j\right)+O(h^4) \tag{2.11}$$

Using $\delta_x^2 u_i=\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}$ into Eq. (2.11), we get

$$(1-6r)u_{i-1}^{j+1}+(10+12r)u_i^{j+1}+(1-6r)u_{i+1}^{j+1}=(1+6r)u_{i-1}^{j}+(10-12r)u_i^{j}+(1+6r)u_{i+1}^{j} \tag{2.12}$$

where $r=\frac{\alpha k}{h^2}$.

Eq. (2.12) is a fourth-order compact finite difference scheme for solving Eq. (2.1) with the prescribed initial and dirichlet boundary conditions.

2.2. Order of the Method

Let us write the difference equation in Eq. (2.12) as

$$L^h u\left(x_i,t_{j+1}\right)=R^h u\left(x_i,t_j\right)$$

where $L^h u(x_i,t_{j+1})$ and $R^h u(x_i,t_j)$ represent the expressions on the left and right hand sides of Eq. (2.12) respectively.

We now plug the exact solution into Eq. (2.12) to obtain

$$\begin{aligned}&L^h u\left(x_i,t_{j+1}\right)-R^h u\left(x_i,t_j\right)\\&=(1-6r)u_{i-1}^{j+1}+(10+12r)u_i^{j+1}+(1-6r)u_{i+1}^{j+1}=(1+6r)u_{i-1}^{j}+(10-12r)u_i^{j}+(1-6r)u_{i+1}^{j}\end{aligned} \tag{2.13}$$

If we expand $L^h u\left(x_i,t_{j+1}\right)$ in terms of the Taylors series in space about the point x_i we have

$$L^h u(x_i, t_{j+1}) =$$
$$12u(x_i, t_{j+1}) + (1-6r)\left[h^2 \frac{\partial^2 u(x_i, t_{j+1})}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 u(x_i, t_{j+1})}{\partial x^4} + \frac{h^6}{360}\frac{\partial^6 u(x_i, t_{j+1})}{\partial x^6} + \ldots\right] \quad (2.14)$$

Expanding Eq. (2.14) using Taylors series in time about t_j gives

$$L^h u(x_i, t_{j+1}) = 12u(x_i, t_j) + 6k\frac{\partial u(x_i, t_j)}{\partial t} + 6k^2 \frac{\partial^2 u(x_i, t_j)}{\partial t^2} + 2k^3 \frac{\partial^3 u(x_i, t_j)}{\partial t^3} +$$
$$(1-6r)\left\{h^2 \frac{\partial^2 u(x_i, t_j)}{\partial x^2} + kh^2 \frac{\partial^3 u(x_i, t_j)}{\partial t \partial x^2} + \frac{k^2 h^2}{2}\frac{\partial^4 u(x_i, t_j)}{\partial t^2 \partial x^2} + \frac{h^4}{12}\frac{\partial^4 u(x_i, t_j)}{\partial x^4} + \frac{h^6}{360}\frac{\partial^4 u(x_i, t_j)}{\partial t^2 \partial x^2} + \ldots\right\} \quad (2.15)$$

Similarly, if we expand $R^h u(x_i, t_j)$ in terms of the Taylors series about the point (x_i, t_j), we get

$$R^h u(x_i, t_j) = 12u(x_i, t_j) + (1+6r)\left\{h^2 \frac{\partial^2 u(x_i, t_j)}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 u(x_i, t_j)}{\partial x^4} + \frac{h^6}{360}\frac{\partial^4 u(x_i, t_j)}{\partial t^2 \partial x^2} + \ldots\right\} \quad (2.16)$$

Subtracting Eq. (2.16) from Eq. (2.15), we get the following expression:

$$L^h u(x_i, t_{j+1}) - R^h u(x_i, t_j)$$
$$= 12k\frac{\partial u(x_i, t_j)}{\partial t} - 12rh^2 \frac{\partial^2 u(x_i, t_j)}{\partial x^2} + 6k^2 \frac{\partial^2 u(x_i, t_j)}{\partial t^2} + kh^2 \frac{\partial^3 u(x_i, t_j)}{\partial t \partial x^2} + 2k^3 \frac{\partial^3 u(x_i, t_j)}{\partial t^3} + \quad (2.17)$$
$$\frac{h^2 k^2}{2}\frac{\partial^4 u(x_i, t_j)}{\partial t^2 \partial x^2} - 6rh^2 \frac{\partial^3 u(x_i, t_j)}{\partial t \partial x^2} - rh^4 \frac{\partial^4 u(x_i, t_j)}{\partial x^4} - \frac{rh^6}{30}\frac{\partial^6 u(x_i, t_j)}{\partial x^6} + \ldots$$

Before simplifying Eq. (2.17), let us make the following relations:

$$u_t = \alpha u_{xx},\ u_{tt} = \alpha u_{xxt} = \alpha (u_t)_{xx} = \alpha^2 u_{xxxx}$$
$$u_{tt} = \alpha u_{xxt} \Rightarrow u_{xxt} = \alpha u_{xxxx}$$
$$h^2 k u_{xxt}(x_i, t_j) = h^2 k \alpha u_{xxxx}(x_i, t_j) = rh^4 u_{xxxx}(x_i, t_j), \quad \alpha k = rh^2$$

Having utilized these relations, Eq. (2.17) will have the following final form:

$$L^h u\left(x_i,t_{j+1}\right)-R^h u\left(x_i,t_j\right)=$$

$$\frac{\partial u\left(x_i,t_j\right)}{\partial t}-\alpha\frac{\partial^2 u\left(x_i,t_j\right)}{\partial x^2}+\frac{k^2}{6}\frac{\partial^3 u\left(x_i,t_j\right)}{\partial t^3}-\frac{\alpha h^4}{360}\frac{\partial^6 u\left(x_i,t_j\right)}{\partial x^6}+\frac{(h^2k-6\alpha k^2)}{24}\frac{\partial^4 u\left(x_i,t_j\right)}{\partial t^2\partial x^2}+\frac{h^2k^2}{6}\frac{\partial^5 u\left(x_i,t_j\right)}{\partial t^3\partial x^2}+\dots \tag{2.18}$$

Since, $\frac{\partial u\left(x_i,t_j\right)}{\partial t}-\alpha\frac{\partial^2 u\left(x_i,t_j\right)}{\partial x^2}=0$, the scheme in Eq. (2.12) for solving Eq. (2.1) is fourth-order convergent in space and second-order convergent in the time variable.

2.3. Stability Analysis

In this section, we present the stability analysis of Eq. (2.12) with Dirichlet boundary conditions using the matrix method. In matrix form, Eq. (2.12) can be written as

$$Au^{j+1}=Bu^j \tag{2.19}$$

Where A and B are matrices of order $N-1$, and given by

$$A=\begin{bmatrix} 10+12r & 1-6r & 0 & 0 & \cdots & 0 \\ 1-6r & 10+12r & 1-6r & 0 & \cdots & 0 \\ 0 & 1-6r & 10+12r & 1-6r & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1-6r & 10+12r & 1-6r \\ 0 & 0 & 0 & 0 & 1-6r & 10+12r \end{bmatrix},$$

$$B=\begin{bmatrix} 10-12r & 1+6r & 0 & 0 & \cdots & 0 \\ 1+6r & 10-12r & 1+6r & 0 & \cdots & 0 \\ 0 & 1+6r & 10-12r & 1+6r & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1+6r & 10-12r & 1+6r \\ 0 & 0 & 0 & 0 & 1+6r & 10-12r \end{bmatrix}$$

For every row, $|10+12r|>|1-6r|+|1-6r|,\ \forall r>0$. Hence, A is invertible.

From Eq. (2.19), we obtain

$$u^{j+1}=A^{-1}Bu^j \tag{2.20}$$

Since A and B are symmetric and commute,

$$D=A^{-1}B \tag{2.21}$$

is real and symmetric.

Let μ_k, γ_k and λ_k are respectively the eigenvalues of A, B, and D. The scheme in Eq. (2.12) is stable if the modulus of every eigenvalues of D is less than unity.

By the idea given in (Smith, 1985 p.156),

$$\mu_k = (10+12r) + 2(1-6r)\cos\frac{k\pi}{N} = 12 - 2(1-6r)\sin^2\frac{k\pi}{2N}$$

$$\gamma_k = (10-12r) + 2(1+6r)\cos\frac{k\pi}{N} = 12 - 2(1+6r)\sin^2\frac{k\pi}{2N}, \quad k = 1(1)N-1.$$

Hence, the eigenvalues of D can be expressed in terms of μ_k and γ_k as:

$$\lambda_k = \frac{\gamma_k}{\mu_k} = \frac{6-(1+6r)\sin^2\frac{k\pi}{2N}}{6-(1-6r)\sin^2\frac{k\pi}{2N}} < 1, \quad \forall r > 0.$$

Thus, the scheme in Eq. (2.12) is unconditionally stable for solving the one-dimensional heat equation with dirichlet boundary conditions.

3. Richardson extrapolation

Richardson extrapolation (RE) is a powerful computational technique that can be implemented in an attempt to increase the accuracy of the numerical solutions of differential equations (Samarskii, 2001). The RE can also be viewed as a technique that accelerates the convergence of the method. The basic concept behind the RE is to solve the difference scheme on two or more consecutive grids and then combine the resulting approximate solutions with appropriate weights to generate higher-order numerical solutions (Merga, F.E, Chemeda, H.M., 2021 and Siraj et al., 2019).

Assume $\Omega_{h,k}$ is the coarser grid and the approximate solution of the expression below is valid.

$$U_h^k = u(x,t) + Ch^4 + R(h,k), \quad (x_i, t_j) \in \Omega_{h,k} \tag{3.1}$$

For the finer grid, $\Omega_{h/2,k}$, assume that the following expression is also valid.

$$U_{h/2}^k = u(x,t) + C\left(\frac{h}{2}\right)^4 + R\left(\frac{h}{2}, k\right), \quad (x_i, t_j) \in \Omega_{h/2,k} \tag{3.2}$$

where $R(h,k)$ and $R(h/2,k)$ are the remainder terms of order $O(h^6, k^2)$ and C is a constant independent of h, and k. To eliminate Ch^4 multiply Eq. (3.1) by $-1/15$ and Eq. (3.2) by $16/15$ and then combine to get a better approximation on the coarser gird $\Omega_{h,k}$.

$$U^{extr}=\frac{16U_{h/2}^{k}-U_{h}^{k}}{15}+O\left(h^{6},k^{2}\right),\ \ \left(x_i,t_j\right)\in\Omega_{h,k} \qquad (3.3)$$

Eq. (3.3) is the Richardson extrapolation technique combined with the scheme in Eq. (2.12) for solving the one-dimensional heat equation in Eq. (2.1) with Dirichlet boundary conditions.

4. Numerical Experiments

To study the performance of the proposed method, we considered two model problems of 1D heat equation with Dirichlet boundary conditions whose exact solution is available.

The numerical results are reported using the maximum error $\|e\|L_\infty$ and the root mean square error L_2 at $T=1$ for different values h are computed using the formulas:

$$\|e\|L_\infty=\max_{1\le i\le N-1}\left(\left|u^{exact}(x_i,1)-u^{approx.}(x_i,1)\right|\right).$$

$$\|e\|L_2=\sqrt{\frac{1}{N}\sum_{i=1}^{N}|u^{exact}(x_i,1)-u^{approx.}(x_i,1)|^2}.$$

Example 1*:* Consider the one-dimensional heat equation (Abdul-Majid, 2009, Hooshmandasl et al., 2012)

$$u_t=u_{xx},\ \ 0<x<1,\ \ 0<t\le T,$$

with an initial condition:

$$u(x,0)=\sin(\pi x),\ \ 0\le x\le 1,$$

and boundary conditions:

$$u(0,t)=u(1,t)=0,\ \ 0\le t\le T.$$

And the exact solution is given by:

$$U(x,t)=\sin(\pi x)\exp(-\pi^2 t).$$

In the subsequent tables of numerical values, FOW-RE is short for ‘fourth-Order scheme with RE’ and FOWO-RE is short for ‘fourth-Order scheme without RE’.

Table 1: Maximum and root mean square errors for test problem 1 at T=1 and Δt=0.0001

h	$\|e\|L_\infty$ -FOW-RE	$\|e\|L_2$ -FOW-RE	$\|e\|L_\infty$ -FOWO-RE	$\|e\|L_2$ -FOWO-RE
1/10	1.25705185e-11	6.13378494e-12	2.07625270e-08	1.39980928e-08
1/20	8.35201813e-12	4.12476771e-12	1.25479361e-09	8.65889852e-10
1/40	8.28850350e-12	4.11859049e-12	3.95158922e-11	2.75990963e-11
1/80	8.28792173e-12	4.13107113e-12	3.63796927e-11	2.55650420e-11

Table 2: Comparison of maximum and mean root square errors for model problem 1 at $h = 0.025$ and $\Delta t = 0.0001$

t	Present Method		Hooshmandasl et al., (2012)	
	$\|e\|L_\infty$ -FOW-RE	$\|e\|L_2$ -FOW-RE	$\|e\|L_\infty$	$\|e\|L_2$
0.1	5.972544e-09	2.967781e-09	6.79e-3	4.86e-3
0.3	2.488959e-09	1.236773e-09	3.76e-4	8.87e-5
0.5	5.762401e-10	2.863360e-10	2.44e-4	1.73e-3
0.7	1.120646e-10	5.568536e-11	3.17e-4	2.04e-4
0.9	2.001475e-11	9.945407e-12	3.14e-3	2.14e-3
1.0	8.288504e-12	4.131071e-12	3.32e-3	2.15e-3

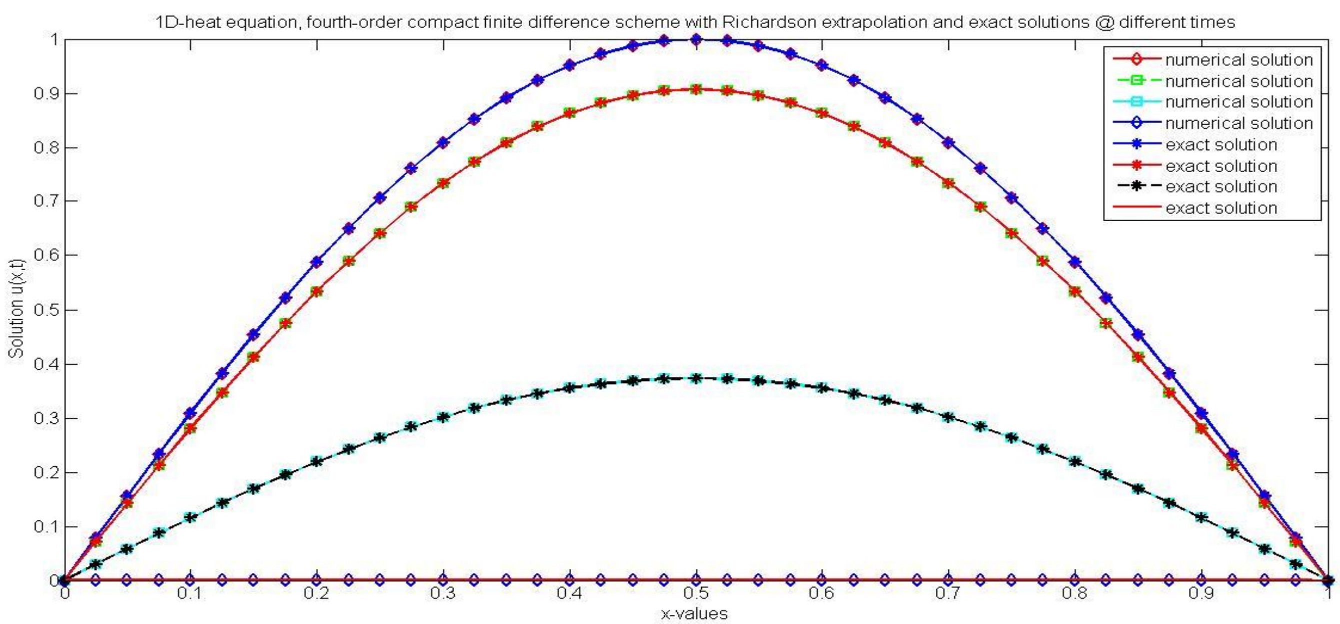

Fig.1 Exact and numerical solutions of model problem 1 at different times

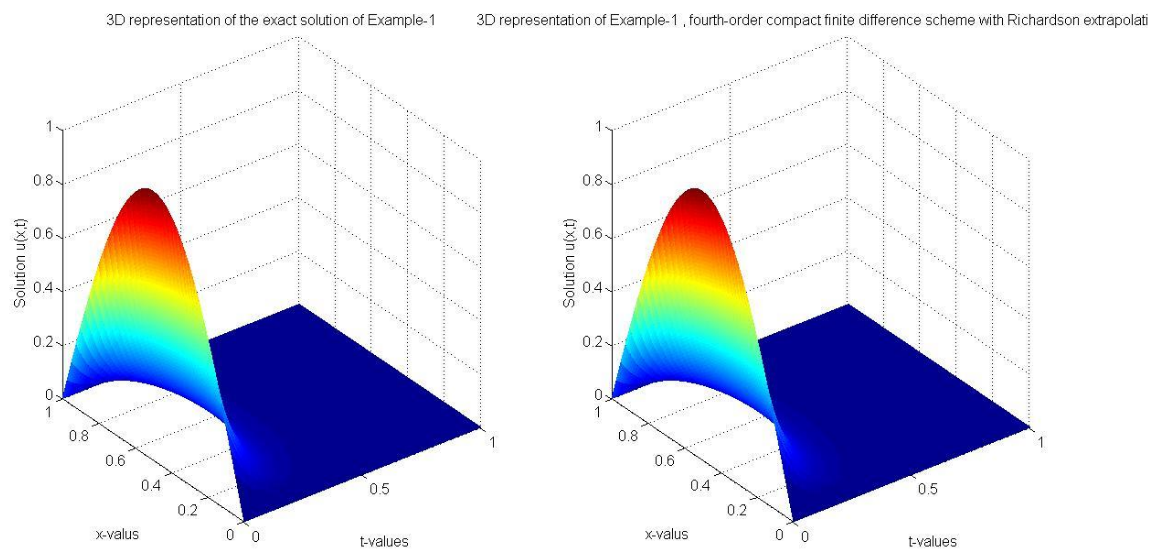

Fig.2 Solution profiles of the exact and approximate solutions of model problem 1

Example 2: Consider the one-dimensional heat equation (Abdul-Majid, 2009, Hooshmandasl et al., 2012)

$$u_t = u_{xx}, \quad 0 < x < 1, \quad 0 < t \le T,$$

whose initial and boundary conditions are respectively given by

$$u(x,0) = \sin(x), \quad 0 \le x \le 1$$
$$u(0,t) = 0, \quad u(1,t) = \sin(1)\exp(-t), \quad 0 \le t \le 1$$

And its exact solution is given by

$$U(x,t) = \sin(x)\exp(-t), \quad 0 \le x \le 1, \quad 0 \le t \le 1$$

Table 3: Comparison of Maximum and root mean square errors for test problem 2 at T=1 and Δt=0.0001

h	$\|e\|L_\infty$-FOW-RE	$\|e\|L_2$ - FOW-RE	$\|e\|L_\infty$-FOWO-RE	$\|e\|L_2$ FOWO-RE
1/10	4.23461266e-12	2.10748968e-12	1.01487874e-08	6.97599928e-09
1/20	4.13763468e-12	2.07310909e-12	6.17913831e-10	4.32712210e-10
1/40	4.00940392e-12	2.02037252e-12	1.95038982e-11	1.38167780e-11
1/80	4.18642898e-12	2.11563374e-12	1.79345427e-11	1.27775937e-11

Table 4: Comparison of maximum and mean root square errors for model problem 2 at $h = 0.025$ and $\Delta t = 0.0001$

t	Present Method		Hooshmandasl et al., (2012)	
	$\|e\| L_\infty$ -FOW-RE	$\|e\| L_2$ -FOW-RE	$\|e\| L_\infty$	$\|e\| L_2$
0.1	5.896283e-12	2.952835e-12	1.07e-5	4.46e-6
0.3	7.521372e-12	3.788042e-12	3.45e-6	2.15e-7
0.5	6.537937e-12	3.295122e-12	5.13e-6	3.18e-6
0.7	5.400957e-12	2.722561e-12	7.45e-6	4.71e-6
0.9	4.428041e-12	2.231319e-12	9.47e-6	6.04e-6
1.0	4.009404e-12	2.020373e-12	1.02e-5	6.55e-6

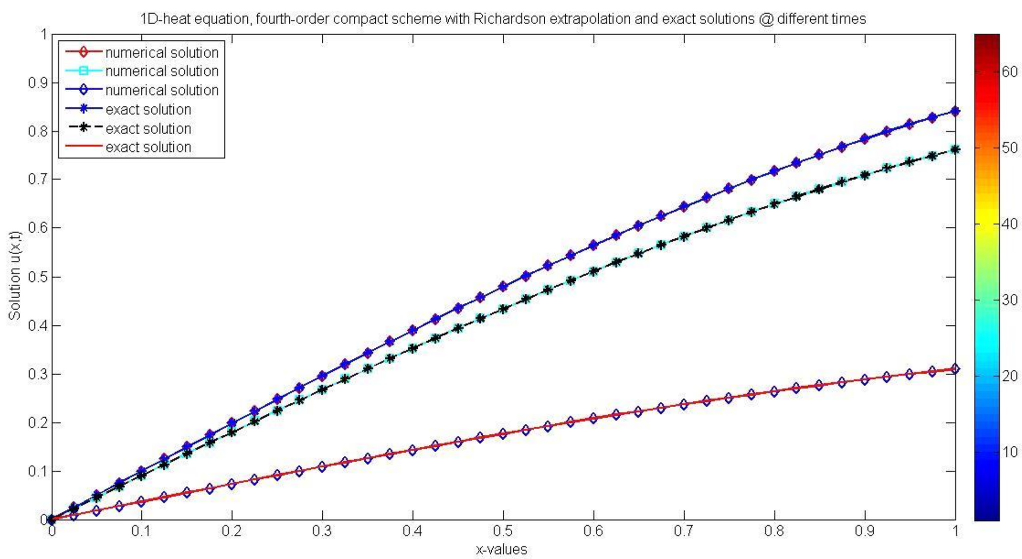

Fig.3 Exact and numerical solutions of model problem 2 at different times

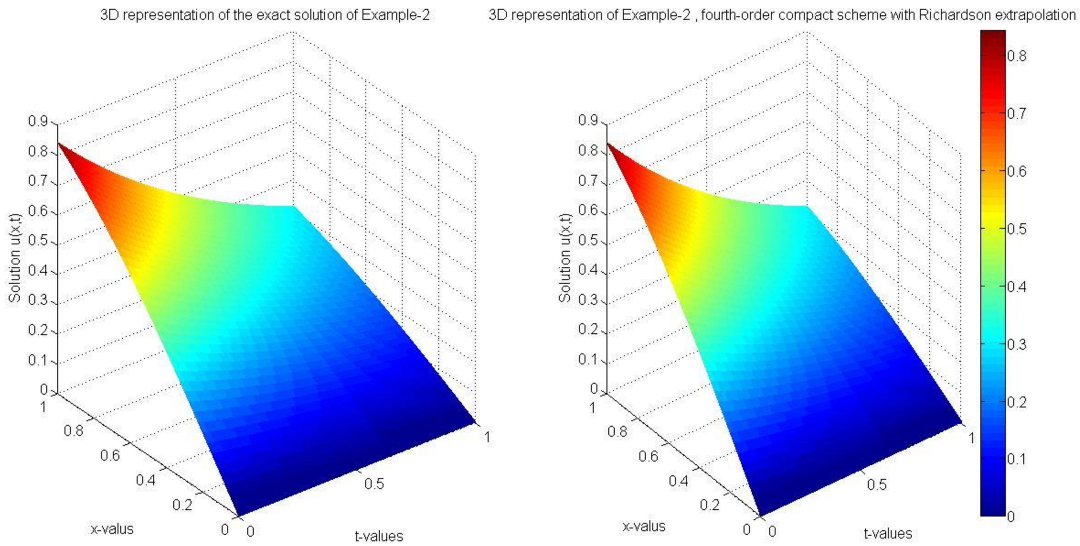

Fig.4 Solution profiles of the exact and approximate solutions of model problem 2

5. DISCUSSION AND CONCLUSION

5.1. DISCUSSION

The numerical results presented in Tables (1, 2, 3 and 4) clearly show that when combined with RE, the proposed method is quite efficient for solving the one-dimensional heat equations. In the absence of the RE, the proposed method attains as accurate results as when it is combined with RE only if further mesh refinement is made which in turn requires extra computational time and memory. For instance, for both model problems, the accuracy attained by FOWO-RE when $h = 1/80$ is still lagging behind the accuracy obtained by FOW-RE even when coarser step lengths such as $h = 1/10$ is used. Besides, the numerical results presented in Tables (1 and 3) and the simulations in Figures (1 and 3) confirm that the numerical solutions are in a good agreement with the exact solution. The 3D simulations in Figures (2 and 4) also show that the application of Richardson extrapolation significantly improves the accuracy of the numerical results. The numerical results in Tables (2 and 4) further confirm that the present method significantly improves the results of Hooshmandasl et al. (2012).

5.2. CONCLUSION

This paper presented fourth-order compact finite difference method combined with Richardson extrapolation for solving the one-dimensional heat equation. The order of convergence and stability conditions of the present method was well established. To illustrate the applicability and efficiency of the method, two model problems are considered and solved for different values of the spatial and temporal step lengths h and k respectively. The numerical solutions are presented using Tables and the simulations to the model problems are demonstrated using Figures generated via MATLAB codes. The proposed method is found to be unconditionally stable, consistent and generates accurate numerical solution for solving the one-dimensional heat equation.

Authors' Contributions

Feyisa Edosa Merga came up with the title and wrote, analyzed and interpreted the data obtained from executing the numerical examples using MATLAB source codes. Hailu Muleta Chemeda conducted the proofs of the order of convergence; stability analysis and wrote a MATLAB code and executed to generate numerical solutions and simulations of the numerical examples. The authors read and agreed on the final version of the manuscript.

Conflicts of interests

We have no competing interests.

REFERENCES

Abdul-Majid, W. (2009): Partial differential equations and Solitary Waves theory, Springer, USA.

Andrea, F. D. and Vautard, R. (2001): Extra tropical low- frequency variability as a low-dimensional problem I: a simplified model. Q.J.R. Meteorol. Soc., 127(54), 1357-1374.

Benyam, M. (2015): Numerical solutions of a one-dimensional heat equation with Dirichlet boundary conditions. American Journal of Applied Mathematics, 3(6), 305-3011.

Carlos, H. M., Leandro, A. N, Casmo, D. S. and Ana, P. da. S. V. (2013): Highly accurate numerical solutions with repeated Richardson extrapolation for 2D Laplace equation. Applied Mathematical Modeling, 37, 7386-7397.

Chen, C., Zhang, X., Liu, Z., and Zhang, Y. (2020): A new high order Compact Finite-Difference Scheme based on precise integration method for the numerical simulation of parabolic equations. Advanced in Numerical Applications, 15, 1-28.

Dabral, V., Kapour, S. and Dhawan, S. (2011): Numerical solutions of one-dimensional heat equation; B-Spline finite element method. Indian Journal of Computer Science & Engineering, 2(2), 222-235.

Dehghan, M. and Tatari, M. (2010): A method for solving partial differential equations via Radial basis functions, Application to the heat equation. Engineering Analysis with boundary element, 34(3), 206-212.

Dimov, I., Kandilarov, J., Todorov, V. and Vulkov, L. (2016): Higher-Order Compact difference schemes with Richardson Extrapolation for semi-linear parabolic systems. AIP Conference Proceedings 1789, 0030002, 1-8.

Gonzalez-Parra, G., Arenas, A. J. and Che-Charpenter, B. M. (2010): Combination of nonstandard schemes & Richardson Extrapolation to improve the numerical solutions of population models. Mathematical & Computer Modeling, 52, 1030-1036.

Hooshmandasl MR., Haydari MH. and Maalek Ghaini FM. (2012): Numerical solutions of one- dimensional heat equations by using Chebysheve Wavelets methods. J Applied Computat Mathemat, 1(6), 1-7.

Liao, H. and Sun, Z. (2011): Maximum norm error estimate of efficient difference schemes for second-order wave equations. J.Comput. Appl. Math., 235, 2217-2233.

Marchuk, G.I. and Shaidurov, V.V. (1983): Difference Methods and their Extrapolations, Springer-Verlag, New York.

Merga, F.E., Chemeda, H.M. Modified Crank–Nicolson Scheme with Richardson Extrapolation for One-Dimensional Heat Equation. *Iran J Sci Technol Trans Sci* (2021).

Necati, O. M., Helcio, R.B. O., Marcelo, J.C. and Renato, M. C. (2017): Finite Difference methods in heat transfer, CRC Press, Taylor & Francis Group.

Olusegun, A. O., Yeak, S. H. and Ezekiel, B. O. (2017): Finite Difference approximations of heat equations via C. Journal of Applied Science & Environmental Sustainability, 3(7), 188-200.

Samarskii, A. A. (2001): The Theory of Difference Schemes, Marcel Dekker, New York.

Siraj, M. K., Duressa, G. F. and Bullo, T. A. (2019): Fourth-order stable central finite difference with Richardson extrapolation method for second-order self-adjoint singularly perturbed boundary value problems. J Egypt Math Soc, 27(50), 1-14.

Smith, G.D. (1985): Numerical solution of partial differential equations, Finite difference methods, Oxford University Press, New York.

Sun, P., LUO, Z. and Zhou, Y. (2010): Some Reduced Finite Difference Schemes based on a proper Orthogonal Decomposition technique for Parabolic Equations. Appl. Numer. Math., 60(1-2), 154-164.

Wang, Y. and Zhang, J. (2009): Sixth-order Compact Scheme Combined with multi-grid method & Extrapolation technique for 2-D Poison Equation. J. Comput. Phys., 228(1), 137-146.

Zhang, J., Geng, X. and Dai, R. (2012): Analysis on two approaches for high order accuracy finite difference computation. Applied Mathematics Letters, 25, 2081-2085.